

ANSWERS

1. A
2. B
3. B
4. D
5. B
6. B
7. E (16807)
8. A
9. D
10. C
11. C
12. C
13. A
14. C
15. D
16. D
17. C
18. A
19. C
20. C
21. A
22. D
23. E (5/7)
24. B
25. B
26. E (3074)
27. A
28. D
29. D
30. D

SOLUTIONS

1. Among the 12 letters there are 3 A's, 2 T's, and 2 H's. Hence, the number of ways is

$$\frac{12!}{3!2!2!} = \frac{12!}{24} = \frac{12!}{4!}.$$

2. The primes one can roll are 2, 3, 5, 7, and 11. There are 1, 2, 4, 6, and 2 ways to roll these, respectively. Out of 36 ways to roll two dice, the probability is

$$\frac{1 + 2 + 4 + 6 + 2}{36} = \frac{15}{36} = \frac{5}{12}.$$

3. Using only one of the three colors, there are 3 ways to color the vertices of a square. Using two colors, say A and B, we can color three of the vertices using A and one using B, or we can color two vertices with A and two with B. For the 3 vertex-1 vertex case, there are three choices of color for three of the vertices and two choices for the single vertex, for 6 ways. In the 2 vertex-2 vertex case, it matters if the vertices colored the same are adjacent or diametrically opposite. If they are adjacent, then there are only 3 ways to color them (since A-A-B-B is the same as B-B-A-A due to rotation). If they are opposite, there are 3 ways to color them. Finally, using all three colors, say A, B, and C, our only option is to color two vertices with the same color and two others each differently. Again, it matters if the vertices colored the same are adjacent or diametrically opposite. If they are adjacent, then there are 3 ways to choose the color for two of the vertices, but we note that A-B-C-C is different from B-A-C-C (they are reflections of each other, but not rotations) so we multiply by 2 to obtain 6 ways in this case. If they are opposite, then there are 3 ways. The total number of ways is therefore $3 + 6 + 3 + 3 + 6 + 3 = 24$.

4. To form three teams of seven, I choose seven students from twenty-one; then I choose seven from the remaining fourteen. This is clearly $\binom{21}{7}\binom{14}{7}$.

5. I can flip exactly three heads in a row by flipping TTHHH, THHHT, HHHTT, HTHHH, or HHHHT. (The problem did not say that the three heads in a row were the only heads that could be flipped.) Hence, out of 32 ways to flip five coins, the probability is $5/32$.

6. The cubic will have exactly three roots if the graph is placed in the coordinate plane so that the x -axis passes through it three times; this occurs if the x -axis is between the polynomial's local minimum value and local maximum value. Since $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, the local extrema are $f(1) = k + 4$ and $f(3) = k$. Hence we require $k \leq 0 \leq k + 4$, and this implies that $-4 \leq k \leq 0$. Thus, k must be chosen from the interval $(-4, 0)$ which has length 4. Out of the possible interval of length 10, the probability is $4/10 = 2/5$.

7. There are seven movies playing, so each friend has seven options. Consequently, the number of ways is $7^5 = 16807$.

8. Since 15,015 factors into $3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, it has only five prime factors. The number of positive integer divisors is $2^5 = 32$, so the probability is $5/32$.

9. Arrange the girls first; there are $7!$ ways to do this. To ensure the boys are not placed next to each other, we place them in between the girls, in front of the first girl, or behind the last girl. There are then 8 places for the first boy, 7 for the second, and 6 for the third. The total is then $7! \cdot 8 \cdot 7 \cdot 6 = 5040 \times 328 = 1693440$.

10. Since all the boys must be together, treat them as one “person” so that there are 8 people to arrange; there are $8!$ ways to do this. However, there are also $3!$ ways to arrange the boys who stand together. Therefore the total number of ways is $8! \cdot 3! = 40320 \times 6 = 241920$.

11. There are $3!$ ways to choose the orders of preference. There are 3 ways to choose the dissenting judge. Once the dissenting judge has an order of preference, there are 5 choices of preferences left for the other two judges. Therefore the total number of ways is $3! \cdot 3 \cdot 5 = 6 \cdot 15 = 90$.

12. First, we find the number of nonnegative integer solutions using the stars-and-bars method. There are twenty stars and two bars (since we need two bars to separate the twenty stars into the categories of x , y , and z). Thus, the number of nonnegative integer solutions is $\binom{20+2}{2} = \binom{22}{2} = 11 \cdot 21 = 231$. To find the number of positive integer solutions, we put three stars into the categories at the start, leaving 17 stars and two bars. Thus, the number of positive integer solutions is $\binom{17+2}{2} = \binom{19}{2} = 19 \cdot 9 = 171$. Finally, the probability is $171/231 = 57/77$.

13. For each combination of letters, there is only one ordering that will satisfy the necessary conditions to be a word. Thus we do not need to worry about ordering the word once we choose the letters. Each letter can be either used or not used, which is 2^{26} possibilities—but if no letter is chosen, there is not a word. Hence, there are $2^{26} - 1$ words.

14. Let S be a musical event and let T be a math tournament. The probability that a day has both is therefore

$$P(S \cup T) = \frac{1}{4} + \frac{1}{3} - \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{6}{12} = \frac{1}{2}.$$

Since $P(S) = 1/4$, we compute

$$P(S | S \cup T) = \frac{P(S)}{P(S \cup T)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

Alternately, we could assume there are 4 tournament days and 3 musical event days. Then there is 1 day on which there is a tournament and a musical event. Hence there are $4 + 3 - 1 = 6$ highlighted days, exactly 3 of which are musical events, for a probability of $1/2$.

15. At least two members must be girls, so we compute the number of ways to form a group with 2 girls, 3 girls, and 4 girls, and add them together. Hence, the total number of ways is

$$\binom{4}{2} \binom{3}{3} + \binom{4}{3} \binom{3}{2} + \binom{4}{4} \binom{3}{1} = 6 \cdot 1 + 4 \cdot 3 + 1 \cdot 3 = 6 + 12 + 3 = 21.$$

16. For Pierre to win, Pierre must roll a 5 or a 6, which will occur with probability $2/6 = 1/3$. However, for Pierre to eventually win on the n th roll, no one can win on the preceding $n - 1$ rolls. No one wins when they roll a 2, 3, or 4, which occurs with probability $3/6 = 1/2$. Thus, the probability Pierre eventually wins is the sum of the infinite geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{3}\right) = \frac{1/3}{1 - 1/2} = \frac{2}{3}.$$

17. The maximum number of intersection points created by n intersecting lines is $\frac{1}{2}n(n - 1)$ and the maximum number of intersection points created by m intersecting circles is $m(m - 1)$. (These can be proved by induction.) The maximum number of intersection points among n lines and m circles is $2mn$, since each line falls across each circle twice. Thus the maximum number of intersection points is $\frac{1}{2}n(n - 1) + m(m - 1) + 2mn$. When $m = n = 10$, this becomes $45 + 90 + 200 = 335$.

18. We use complementary counting: we count the number of four-digit integers that have no repeated digits, and subtract that from 9000, the number four-digit positive integers. If we want no repeated digits, then the thousands digit can be any of the digits from 1 to 9. The hundreds digit can be any of the digits from 0 through 9 except the digit in the thousands place; hence 9 possible digits. The tens place can be any of the digits from 0 through 9 except those in the thousands and hundreds places; hence 8 possible digits. Similarly, there are 7 possible units place digits. The number of four-digit integers with no repeated digits is $9 \cdot 9 \cdot 8 \cdot 7 = 81 \times 56 = 4536$. Therefore the probability that a four-digit positive integer has a repeated digit is

$$\frac{9000 - 4536}{9000} = \frac{4464}{9000} = \frac{62}{125}.$$

19. Suppose we have such a five-digit integer where the digits are strictly in non-increasing order. Let this number have the representation $abcde$ where a, b, c, d , and e are digits. Then, since each digit must be no greater than the digit that comes before it, we have, for instance, $d + 1 > e$. Likewise, we have $c + 1 > d$, $b + 1 > c$, and $a + 1 > b$. Noting that e could be zero and a could be 9, we have

$$13 \geq a + 4 > b + 3 > c + 2 > d + 1 > e \geq 0.$$

However, to find integers $a + 4, b + 3, c + 2, d + 1$, and e which satisfies the inequalities above implies that we want to choose five integers from among the integers 0 through 13 which are in strictly decreasing order. This can be done by simply choosing five integers from the 14 integers from 0 to 13 (since there is only one way to arrange the integers in decreasing order), so we have

$$\binom{14}{5} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{5 \cdot 4 \cdot 3 \cdot 2} = 14 \cdot 13 \cdot 11 = 2002.$$

However, this count includes the five-digit "number" 00000. Thus there are 2001 five-digit integers.

(Note: if we choose 9, 4, 7, 12, and 6, and put them decreasing order, we have 12, 8, 7, 6, 4. Then since $a + 4 = 12$, we have $a = 8$; since $b + 3 = 8$, we have $b = 5$; and so on. This selection of integers corresponds to the five-digit integer 85554.)

20. Of the three-digit squares, we can only form those that do not have a 7, 8, 9, or 0. This leaves 121, 144, 225, 256, 324, 361, 441, and 625 as the only perfect squares that could be formed by rolling a die. Of the 216 ways to roll a die three times, the probability we get a perfect square is $8/216 = 1/27$.

21. The region bounded by the coordinate axes (since the numbers must be positive) and the line $x + y = 4$ is an isosceles right triangle of area 8. The region bounded by the coordinate axes and the line $x + y = 2$ is an isosceles right triangle of area 2 within the larger triangle. Hence the probability is $2/8 = 1/4$.

22. Of the 64 unit cubes, there are 8 with 3 painted faces (the corners of the original cube), there are $2 \cdot 12 = 24$ with 2 painted faces (the edges, not counting the corners), there are $4 \cdot 6 = 24$ with exactly 1 face painted (the interior of the faces of the original cube), and there are $64 - 8 - 24 - 24 = 8$ with no faces painted. Each face of each unit cube has probability $1/6$ of showing face up when rolled. Therefore, the probability of showing a painted face when rolled is

$$0 \cdot \frac{8}{64} + \frac{1}{6} \cdot \frac{24}{64} + \frac{2}{6} \cdot \frac{24}{64} + \frac{3}{6} \cdot \frac{8}{64} = \frac{1}{16} + \frac{2}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}.$$

23. Let p be the probability that Blaise wins. There is a $1/6$ probability that Blaise and Pierre will roll the same number, in which case Pierre wins. If they roll different numbers, then there is a probability of $1/2$ that Blaise's number is higher since the dice are fair. Thus the probability Blaise wins on the first roll is $(5/6) \cdot (1/2) = 5/12$, and the probability Pierre will win is $1/6$. The probability they will roll again is also $5/12$; if they play again, then Blaise will win with probability p . Therefore

$$p = \frac{5}{12} \cdot 1 + \frac{5}{12} \cdot p + \frac{1}{6} \cdot 0,$$

Which implies $(7/12)p = 5/12$ so that $p = 5/7$.

24. The probability that Zarek does not tie his shoes together correctly on day n is

$$1 - \frac{7}{7n+9} = \frac{7n+2}{7n+9}.$$

It follows that the probability Zarek does not tie his shoes for days 1 through 100 is the product

$$\prod_{n=1}^{100} \frac{7n+2}{7n+9} = \frac{9}{16} \cdot \frac{16}{23} \cdot \frac{23}{30} \cdot \frac{30}{37} \cdots \frac{702}{709} = \frac{9}{709}.$$

25. There are $2 \cdot 4 = 8$ handshakes between each Republican and each Independent. There are $2 \cdot 3 = 6$ handshakes between each Democrat and each Independent. Among the Independents, there is 1 handshake. Among Democrats, there are $3 \cdot 2/2 = 3$ handshakes. Among Republicans, there are $4 \cdot 3/2 = 6$ handshakes. The total number of handshakes is therefore $8 + 6 + 1 + 3 + 6 = 24$.

Alternately, we could use complementary counting: There are $9 \cdot 8/2 = 36$ handshakes between all members, and, if Democrats and Republicans shook hands, there would be $4 \cdot 3 = 12$ handshakes. Hence, there are $36 - 12 = 24$ handshakes without Democrats and Republicans shaking each other's hands.

26. Let p be the probability that Chauncey makes one free-throw out of one attempt. Then the probability that he will make n free-throws out of 2018 is

$$\binom{2018}{n} p^n (1-p)^{2018-n}.$$

We are told that the probability of making 963 out of 2018 free-throws is the same as the probability of making 964 out of 2018 free-throws. Thus, we solve the following equation.

$$\binom{2018}{963} p^{963} (1-p)^{2018-963} = \binom{2018}{964} p^{964} (1-p)^{2018-964}$$

$$\binom{2018}{963} p^{963} (1-p)^{1055} = \binom{2018}{964} p^{964} (1-p)^{1054}$$

$$\frac{2018!}{963! 1055!} (1-p) = \frac{2018!}{964! 1054!} p$$

$$964(1-p) = 1055p$$

$$964 = 2019p$$

Therefore, $p = 964/2019$, so that not making a free-throw is $1 - 964/2019 = 1055/2019$. Finally, the answer is $1055 + 2019 = 3074$.

27. The number of ways of selecting 2 rows from 4 rows is $\binom{4}{2} = 6$. So the 4-by-6 array can be colored to give a rectangle-free coloring as shown below.

Y	Y	Y	G	G	G
Y	G	G	Y	Y	G
G	Y	G	Y	G	Y
G	G	Y	G	Y	Y

No larger grid can be so colored: each column must have 2 dots of each color. Let us justify this statement: Suppose a column has four dots of the same color. If it has one column of 4 green dots then no other column can contain more than one green dot. Therefore more than 2 columns with more than 3 yellow dots implies a yellow rectangle for $n \geq 3$; hence no column has all dots the same color unless $n \leq 2$.

Now suppose that no column has 4 dots colored the same, but that one contains 3 green dots. No other column can contain 3 green dots; at most 3 other columns can contain 2 green dots; no other column is possible without forming a yellow rectangle. Hence $n \leq 4$.

Hence, each column has two dots of each color, giving $n = 6$.

28. Suppose that Zechariah is the $(k + 1)$ th person Jedidiah asks. Since any ordering of the k people before Zechariah is equally likely, and Zechariah is the last in the alphabet, the probability that they are in order is $1/k!$. Since all values of k between 0 and 99 are equally likely,

$$p = \frac{1}{100} \sum_{k=0}^{99} \frac{1}{k!}.$$

Therefore,

$$\frac{e}{p} = 100 \frac{\sum_{k=0}^{\infty} \frac{1}{k!}}{\sum_{k=0}^{99} \frac{1}{k!}}.$$

However, the fractional quantity is very close to 1. (Indeed, it is less than $1/100!$ away from 1.) Thus the answer is 100.

29. We use the principle of inclusion-exclusion. Let the pairs of socks be designated 1, 2, 3 and 4. Let A_i be the set of arrangements where pair i is together, for $i = 1, 2, 3, 4$. Let U be the set of all arrangements of the 8 socks. The total number of arrangements of the 8 socks, allowing a sock to be next to its mate, is $N(U) = 8!/2^4$, where we divide by 2^4 because within each pair the socks can swap places without changing the arrangement. By PIE, we compute the following expression.

$$N(U) - \sum_{i=1}^4 N(A_i) + \sum_{i \neq j} N(A_i \cap A_j) - \sum_{i \neq j \neq k} N(A_i \cap A_j \cap A_k) + N(A_1 \cap A_2 \cap A_3 \cap A_4)$$

We have $N(U)$. Note that $N(A_1) = 7!/2^3$, since we want sock pair 1 to stay together. However, there is nothing special about sock pair 1—any of the sock pairs would give the same number. Hence, $\sum N(A_i) = 4 \cdot 7!/2^3$. We also have $N(A_1 \cap A_2) = 6!/2^2$, since we want to keep pair 1 together and pair 2 together—but again, any combination of two sock pairs would give the same number. Since there are 6 possible ways to choose two pairs from four, $\sum N(A_i \cap A_j) = 6 \cdot 6!/2^2$. Keeping three sock pairs together gives us $\sum N(A_i \cap A_j \cap A_k) = 4 \cdot 5!/2$. Finally, keeping all four sock pairs together is simply $4!$. Putting all of this together, we have that the number of ways I can arrange my socks if no sock is allowed to be next to its mate is

$$\frac{8!}{2^4} - 4 \cdot \frac{7!}{2^3} + 6 \cdot \frac{6!}{2^2} - 4 \cdot \frac{5!}{2} + 4! = 2520 - 2520 + 1080 - 240 + 24 = 864.$$

30. We begin with the number of ways for Robert, Roberto, and Roberta to make their 3×3 squares. There are nine 1×1 squares, so there appear to be 3^9 ways for each of them to make their 3×3 squares. However, this overcounts the true number of ways due to rotations. So let N be the number of distinct ways to arrange nine 1×1 squares into a 3×3 square. Now suppose we have all those distinct ways, and we rotate each one of them 90° , 180° , and 270° . We may organize this into N rows of four copies of each distinct pattern, for a total of $4N$ patterns. Clearly, $4N$ is larger than 3^9 , since there are rotations involved which make some of those appearing in our N rows identical. Let's count how many are identical. First, let's remove the 3^9 possibilities from our list. This makes many rows empty, but those fixed by a rotation remain; that is, those which are identical upon rotation remain. Those fixed by a 90° rotation would be those whose four corners squares are the same color and whose four middle squares are the same color. Note that the center square may be any color. This can happen in 3^3 ways. Also, if a pattern is fixed by a 90° rotation, it is fixed by a 270° rotation, so there are another 3^3 identical patterns. Finally, a 3×3 square is fixed by a 180° rotation if the top row of three squares is the same as the bottom row of three squares, and the two squares on the ends of the middle row are the same color (again, the center square may be any color). This happens in 3^5 ways. So we have a total of $3^9 + 3^3 + 3^3 + 3^5$ patterns in our N rows of 4 rotations. Hence,

$$4N = 3^9 + 3^3 + 3^3 + 3^5 = 27(729 + 2 + 9) = 27 \cdot 740,$$

so that $N = 27 \cdot 185 = 4995$. Thus there are 4995 distinct ways to create a 3×3 square from nine 1×1 squares using three colors.

Now we determine how many ways they can put their 3×3 squares together to make a quilt. Since Bob must make a pink square, there are 4995^3 ways for the others to include their squares. But, as before, this overcounts the real number due to rotations. Thus, we proceed in the same way as before. Let us assume that there are Q distinct ways to put the four 3×3 squares together to make a 6×6 quilt. Now list all Q of them, and rotate them 90° , 180° , and 270° . This creates Q rows of

four, for a total of $4Q$ patterns. Now we count the ones fixed by a rotation. First remove the 4995^3 possibilities from our list. The only ones left are those that are fixed by rotations. The only one fixed by a 90° rotation is the quilt made from all pink 3×3 squares. As this is also fixed by a 270° rotation, there are 2 copies. The only ones fixed by a 180° rotation are those with two pink squares diagonally opposite and two identically patterned squares. This happens in 4995 ways (since we can pick any of the 4995 patterns to duplicate, and one other must be pink).

Now we find Q , the number of ways to create the 6×6 quilt. We have $4Q = 4995^3 + 2 + 4995$ which implies

$$Q = \frac{4995^3 + 4995 + 2}{4} = \frac{(4995 + 1)(4995^2 - 4995 + 2)}{4} = 1249(4995^2 - 4995 + 2).$$

Note that $4995^2 - 4995$ must have a units digit of 0, so $4995^2 - 4995 + 2$ has a units digit of 2. Multiplying this by the units digit of 1249 results in a units digit of 8. Hence, the remainder when Q is divided by 10 is 8.